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Oscillations in double-diffusive convection

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We have studied the transition between oscillatory and steady convection in a simplified model of two-dimensional thermosolutal convection. This model is exact to second order in the amplitude of the motion and is qualitatively accurate for larger amplitudes. If the ratio of the solutal diffusivity to the thermal diffusivity is sufficiently small and the solutal Rayleigh number, R_S, sufficiently large, convection sets in as overstable oscillations, and these oscillations grow in amplitude as the thermal Rayleigh number, R_T, is increased. In addition to this oscillatory branch, there is a branch of steady solutions that bifurcates from the static equilibrium towards lower values of R_T; this subcritical branch is initially unstable but acquires stability as it turns round towards increasing values of R_T. For moderate values of R_S the oscillatory branch ends on the unstable (subcritical) portion of the steady branch, where the period of the oscillations becomes infinite. For larger values of R_S a birfurcation from symmetrical to asymmetrical oscillations is followed by a succession of bifurcations, at each of which the period doubles, until the motion becomes aperiodic at some finite value of R_T. The chaotic solutions persist as R_T is further increased but eventually they lose stability and there is a transition to the stable steady branch. These results are consistent with the behaviour of solutions of the full two-dimensional problem and suggest that period-doubling, followed by the appearance of a strange attractor, is a characteristic feature of double-diffusive convection.

1. Introduction

Geophysics and astrophysics provide many examples of convective phenomena where motion first appears as overstable oscillations. One such example is a fluid layer with a dissolved solute. When the solute gradient is stable, any perturbation to the static equilibrium leads to transient oscillations. It is, however, possible to maintain oscillatory motion at some finite amplitude by heating the layer from below. If the rate of heating is increased the oscillations grow in amplitude until they are superseded by regular convection. This pattern of behaviour occurs only if the solute diffuses less rapidly than heat and is characteristic of double-diffusive convection. Similar effects can be produced with salt and sugar in the laboratory, with heat and salt in the ocean, or with heat and helium in the interior of a star. More complicated double-diffusive

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phenomena appear if the destabilizing thermal gradient is opposed by the effects of a magnetic field or rotation.

Most theoretical studies of double-diffusive convection have concentrated on the linearized equations. Nonlinear theory is more complicated: in certain parameter ranges oscillatory and steady convection coexist and the transition from one to the other has proved difficult to understand. Our aim is to study a system in which this transition can be explored in detail. For this purpose we confine our attention to thermosolutal convection, which provides both the simplest and the most thoroughly investigated example of double-diffusive behaviour (see, for example, Spiegel 1972, Turner 1973, Schechter, Velarde & Platten 1974, Huppert 1977).

The transition from oscillatory to steady motion has been followed in a series of numerical experiments on two-dimensional thermosolutal convection (Huppert 1976, 1977; Huppert & Moore 1976). For moderate values of the solute gradient, measured by a solutal Rayleigh number R_S, the oscillations are always symmetrical, with no distinction between opposite senses of motion; as the thermal gradient, measured by a thermal Rayleigh number R_T, is increased the oscillations keep growing until they abruptly disappear and the solution jumps to a steadily convecting state. With more extreme values of R_S Huppert & Moore find that the pattern changes: the symmetrical oscillations bifurcate to asymmetrical oscillations, with one or other sense preferred. Then, as R_T is increased still further, successive oscillations cease to be identical and the motion becomes aperiodic before jumping to a steady state. Huppert (1976) contrasted this transition with various hypothetical descriptions of the onset of turbulence.

One possibility, which has recently attracted much attention (see, for example, Pomeau 1977, Feigenbaum 1978, 1979, Shimada & Nagashima 1978, Shimizu & Morioka 1978, Curry 1978, Robbins 1978, Ito 1979, Franceschini & Tebaldi 1979, Boldrighini & Franceschini 1979, Coullet, Tresser & Arnéodo 1979, Marzec & Spiegel 1980, Franceschini 1980), is that the transition to aperiodic behaviour proceeds by a sequence of bifurcations, at each of which the period doubles, until the solution becomes chaotic, with the appearance of a strange attractor at some finite value of R_T. Unfortunately, it is not yet possible to solve the relevant partial differential equations with sufficient accuracy to determine whether such a sequence of bifurcations actually occurs for thermosolutal convection. We shall therefore follow a different approach. Instead of grappling with the full problem we take a simplified model, put forward by Veronis (1965), which, as we shall see, reproduces qualitative features of the full system with remarkable fidelity. This model problem, consisting of five coupled nonlinear ordinary differential equations, can easily be solved with sufficient accuracy by a combination of analytical and numerical techniques. In an appropriate parameter range we are indeed able to identify successive bifurcations at which the period doubles (and quadruples, etc.) before the solutions become aperiodic.

When $R_S = 0$ our fifth-order system decouples; the resulting third-order system is identical with the Lorenz (1963) model, which can be derived from the partial differential equations that describe two-dimensional Rayleigh–Bénard convection. The Lorenz equations exemplify the dangers of representing nonlinear problems by drastically truncated systems. They possess steady solutions (corresponding to steady convection) which become unstable to oscillatory perturbations when R_T exceeds a critical value $R_T^{(L)}$. A stable strange-attracting set appears at a value of R_T slightly less than $R_T^{(L)}$.

The Hopf bifurcation at $R_T = R_T^{(L)}$ is subcritical (McLaughlin & Martin 1975; Marsden & McCracken 1976; Robbins 1977); thus there is a sudden transition from a steady solution to aperiodic oscillations as R_T is increased through $R_T^{(L)}$. For $R_T \gg R_T^{(L)}$, on the other hand, the solutions exhibit symmetric oscillations (Robbins 1978); if R_T is now gradually decreased there is first a bifurcation to asymmetric oscillations (Shimada & Nagashima 1978; Shimizu & Morioka 1978; Robbins 1978) which is followed by an infinite sequence of bifurcations, at each of which the period doubles, leading to chaotic behaviour and the appearance of a strange attracting set beyond the accumulation point. As R_T is further reduced, more complicated periodic solutions appear, followed by another transition to chaos and so forth. Thus the Lorenz system provides an excellent paradigm for studying period doubling and the appearance of a strange attractor. However, although these equations do describe the behaviour of certain real systems (e.g. Malkus 1972, Haken 1975), their solutions differ from those of the two-dimensional problem from which they were originally derived. Curry (1978) has developed a fourteenth-order system which approximates more closely to the full problem: he finds that the value of the Rayleigh number at the Hopf bifurcation (corresponding to $R_T^{(L)}$) is increased and that the bifurcation is now supercritical. For R_T slightly greater than $R_T^{(L)}$ there are finite-amplitude oscillations about the steady solutions; as R_T is increased there are further bifurcations leading to period doubling, a torus and, probably, a strange attractor. Accurate solutions of the partial differential equations, obtained using finite differences, also show oscillations about the steady state (Moore & Weiss 1973); these oscillations persist as R_T is increased but neither period doubling nor chaotic behaviour have been detected. On the other hand, the correspondence between the solutions of our fifth-order system and the results of the numerical experiments of Huppert & Moore is so close that period doubling must, we believe, be a feature of the full problem too. Taken together, these computations therefore provide the best available evidence for the appearance of a strange attractor in a finite fluid system.

In this paper we first describe the model problem and summarize the results of linear and finite-amplitude theory. We then investigate steady convection and its stability. The principal results are found in § 5, where we present the oscillatory solutions. For moderate values of R_S, the oscillatory branch (defined by the amplitude as a function of R_T) terminates on the unstable steady branch, where the period of oscillations becomes infinite; for larger R_S, the oscillatory branch typically reaches a maximum before the symmetrical oscillations bifurcate to asymmetrical solutions. Then, as R_T is increased still further, successive oscillations cease to be identical. The solution, however, remains periodic and repeats itself after every second oscillation so that its period is now twice the period of the original solution. Beyond the next transition, the solution repeats itself after every fourth oscillation, and so on. We have been able to identify doubling, quadrupling and octupling of the period before the solutions become aperiodic. Eventually the aperiodic oscillations lose stability and the solution settles on to the steady branch. Many of these features have also been found in a parallel study of convection in a magnetic field (Knobloch, Weiss & Da Costa 1981).

2. The model

We consider two-dimensional convection in a horizontal layer of fluid confined between the planes $z = 0$ and $z = h$. In the Boussinesq approximation the density is taken to be $\rho = \rho_0[1-\alpha T+\beta S]$, where T is the temperature and S the solute density and $\alpha, \beta > 0$. Following Huppert & Moore (1976), we introduce a stream function $\Psi(x,z)$ so that the velocity

$$\mathbf{u} = (-\partial_z\Psi, 0, \partial_x\Psi) \tag{1}$$

and set

$$T = T_0+\Delta T[1-z/h+\Theta(x,z)], \quad S = S_0+\Delta S[1-z/h+\Sigma(x,z)]. \tag{2}$$

We express all variables in dimensionless form, measuring lengths and time in terms of h and h^2/κ_T, respectively. The governing equations can then be written as

$$\sigma^{-1}[\partial_t\nabla^2\Psi+J(\Psi,\nabla^2\Psi)] = R_T\partial_x\Theta-R_S\partial_x\Sigma+\nabla^4\Psi, \tag{3}$$

$$\partial_t\Theta+J(\Psi,\Theta) = \partial_x\Psi+\nabla^2\Theta, \tag{4}$$

$$\partial_t\Sigma+J(\Psi,\Sigma) = \partial_x\Psi+\tau\nabla^2\Sigma, \tag{5}$$

where

$$\sigma = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_S}{\kappa_T}, \quad R_T = \frac{g\alpha\Delta T h^3}{\kappa_T\nu}, \quad R_S = \frac{g\beta\Delta S h^3}{\kappa_T\nu}. \tag{6}$$

Here κ_T, κ_S and ν are the thermal, solutal and viscous diffusivities and the other symbols have their usual meanings; note, however, that we do not exclude the possibility that $\tau > 1$. We restrict attention to the region $\{0 < x < \lambda;\ 0 < z < 1\}$ and adopt the simplest boundary conditions, so that

$$\left.\begin{array}{lllll}\Psi = 0, & \partial_z^2\Psi = 0, & \Theta = 0, & \Sigma = 0 & (z = 0,1),\\ \Psi = 0, & \partial_x^2\Psi = 0, & \partial_x\Theta = 0, & \partial_x\Sigma = 0 & (x = 0,\lambda).\end{array}\right\}. \tag{7}$$

Next we suppose that the variables are expanded in Fourier series and adopt a minimal representation (Veronis 1965) by setting

$$\Psi = 2(2p)^{\frac{1}{2}}\frac{\lambda}{\pi}\sin\frac{\pi x}{\lambda}\sin\pi z\, a(t^*), \tag{8}$$

$$\Theta = 2\left(\frac{2}{p}\right)^{\frac{1}{2}}\cos\frac{\pi x}{\lambda}\sin\pi z\, b(t^*)-\frac{1}{\pi}\sin 2\pi z\, c(t^*), \tag{9}$$

$$\Sigma = 2\left(\frac{2}{p}\right)^{\frac{1}{2}}\cos\frac{\pi x}{\lambda}\sin\pi z\, d(t^*)-\frac{1}{\pi}\sin 2\pi z\, e(t^*), \tag{10}$$

where $t^* = pt$ and $p = \pi^2(1+1/\lambda^2)$. Substituting these five modes into the equations and neglecting all higher harmonics generated by their interactions, we obtain the equations

$$a' = \sigma[-a+r_T b-r_S d], \tag{11a}$$

$$b' = -b+a(1-c), \tag{11b}$$

$$c' = \varpi[-c+ab], \tag{11c}$$

$$d' = -\tau d+a(1-e), \tag{11d}$$

$$e' = \varpi[-\tau e+ad], \tag{11e}$$

where

$$\varpi = \frac{4\pi^2}{p}, \quad r_T = \frac{\pi^2}{\lambda^2 p^3} R_T, \quad r_S = \frac{\pi^2}{\lambda^2 p^3} R_S. \tag{12}$$

These equations are the same as those obtained by Veronis (1965); Siegmann & Rubenfeld (1975) have discussed a loop model of thermohaline convection described by equations (11) with $\varpi = 1$ ($\lambda = 3^{-\frac{1}{2}}$). The system possesses an important symmetry, for it is invariant under the transformation that reverses the signs of a, b and d while leaving c and e unaltered. Moreover, all solutions are uniformly bounded in time.

This model has several appealing properties. There is a static solution whose linear stability is identical to that of the static solution to the full problem. Moreover, results obtained from modified perturbation theory to second order are also identical. Equation (9) includes a limited description of the build-up of the temperature gradient near the boundaries which ensures the existence of a finite solution for finite r_T, while (10) provides for a similar build-up of the gradient of solute concentration which allows the possibility of subcritical convection. For a more careful justification of this approach in a related context, see Knobloch *et al.* (1981).

3. Bifurcations from the static solution

The model equations admit the trivial static solution $a = b = c = d = e = 0$, whose stability has often been discussed (Stern 1960; Veronis 1965; Baines & Gill 1969; Huppert & Moore 1976). Equations (11) linearized about this solution admit solutions that vary as $\exp st^*$. At

$$r_T^{(e)} = 1 + \tau^{-1} r_S, \tag{13}$$

one of the eigenvalues s is zero (corresponding to a marginally stable solution), while at

$$r_T^{(o)} = 1 + (1 + \sigma + \tau)\frac{\tau}{\sigma} + \left(\frac{\sigma + \tau}{\sigma + 1}\right) r_S, \tag{14}$$

there is a pair of pure imaginary eigenvalues $\pm i\omega_0$ (corresponding to the onset of overstability), provided that

$$\omega_0^2 = -\tau^2 + \left(\frac{1-\tau}{1+\sigma}\right)\sigma r_S > 0. \tag{15}$$

Since (15) is equivalent to the condition

$$r_T^{(e)} - r_T^{(o)} = \omega_0^2 \Delta / \sigma\tau > 0, \tag{16}$$

where $\Delta = 1 + \sigma + \tau$, we may distinguish the following two cases:

(i) $\omega_0^2 < 0$. In this case as r_T is increased the static solution loses stability at $r_T^{(e)}$, where there is a bifurcation to a triplet of steady solutions, one of which is the now unstable static solution. The other two are finite-amplitude solutions, differing only in the sign of a, b and d. We may investigate the behaviour of the two branches of steady solutions in the neighbourhood of $r_T^{(e)}$ in terms of the a-mode by setting

$$r_T = r_T^{(e)} + r_2^{(e)} a^2 + O(a^4).$$

From equations (11) it then follows that

$$\sigma\tau^3 r_2^{(e)} = \sigma\tau^3 + \sigma(\tau^2 - 1) r_S = -\tau^2\Delta - (1+\sigma)(1+\tau)\omega_0^2 \tag{17}$$

(cf. Huppert & Moore 1976). When $r_2^{(e)} > 0$ the bifurcation is supercritical and the two steady states are stable, while for $r_2^{(e)} < 0$ the bifurcation is subcritical and the two steady states are unstable with one (small) positive eigenvalue (Hopf 1942; Sattinger 1973; Marsden & McCracken 1976). In either case the static solution loses stability to a direct mode. (We follow here the terminology described by Spiegel (1972).) Nothing special occurs at $r_T^{(o)}$.

(ii) $\omega_0^2 > 0$ $(r_T^{(o)} < r_T^{(e)})$. Here the first bifurcation is a Hopf bifurcation (e.g. Marsden & McCracken 1976) at $r_T = r_T^{(o)}$ and the static solution loses stability to an overstable mode. As r_T is increased towards $r_T^{(e)}$ the pair of imaginary eigenvalues acquire positive real parts and their imaginary parts decrease until they become real and equal (Baines & Gill 1969). Thereafter one of them decreases and passes through zero at $r_T = r_T^{(e)}$, where there is another bifurcation. Since $r_2^{(e)} < 0$, this bifurcation results for $r_T < r_T^{(e)}$ in two finite-amplitude steady solutions in the neighbourhood of the static solution, both of which are unstable with one (large) positive eigenvalue. The small eigenvalue that is present as a result of the proximity to the bifurcation point is now negative (cf. Hopf 1942).

The behaviour of the branch of oscillatory solutions in the neighbourhood of $r_T^{(o)}$ was investigated by Huppert & Moore (1976), who showed that the bifurcation could be subcritical. The required lengthy calculation can be considerably simplified by considering equations (11) rather than the full partial differential equations, by adopting complex notation, and by using the results

$$\sigma r_S = (\omega_0^2+\tau^2)(\sigma+1)/(1-\tau), \quad \sigma r_T^{(o)} = (\omega_0^2+1)(\sigma+\tau)/(1-\tau) \tag{18}$$

to express everything in terms of the oscillation frequency ω_0. The procedure is described in more detail by Knobloch *et al.* (1981). The transition from a supercritical bifurcation to a subcritical bifurcation occurs when $r_2^{(o)} = 0$, which reduces to the quadratic equation

$$4\sigma\omega_0^4 - \omega_0^2[\tau\Delta(4+2\varpi+\varpi^2)+2\varpi\sigma(\Delta+\sigma\tau+\tau^2)] - \varpi^2\Delta\tau(\Delta+\sigma\tau+\tau^2) = 0. \tag{19}$$

This result was obtained by Rubenfeld & Siegmann (1977) for $\varpi = 1$ and the predictions of (19) agree with the numerical results of Huppert & Moore for $\varpi = \frac{8}{3}$. Observe that there is always a positive root for ω_0^2, so that subcritical oscillations will occur for r_S greater than the value corresponding to this root. Finally, we note that all results in this section hold both for the full partial differential system and for the simplified model.

4. Properties of the steady branch

Since

$$\frac{\partial a'}{\partial a}+\frac{\partial b'}{\partial b}+\frac{\partial c'}{\partial c}+\frac{\partial d'}{\partial d}+\frac{\partial e'}{\partial e} = -[\sigma+(1+\varpi)(1+\tau)] < 0, \tag{20}$$

equations (11) define a contraction mapping in a five-dimensional phase space. Solutions must therefore be attracted to a set of zero measure in the phase space; in particular, they may be attracted to a fixed point, a limit cycle or a strange attractor. In this section we discuss nontrivial fixed points; limit cycles and strange attractors follow in § 5.

The time-independent solution to equations (11) is given by

$$r_T = (1+a^2)\left[1+\frac{\tau r_S}{\tau^2+a^2}\right], \tag{21}$$

which is a quadratic equation for $a^2(r_T)$ (Veronis 1965). For $r_2^{(e)} < 0$ there exists a minimum Rayleigh number for steady convection given by

$$r_T^{(\mathrm{min})} = [(1-\tau^2)^{\frac{1}{2}} + (\tau r_S)^{\frac{1}{2}}]^2; \tag{22}$$

from (14) and (22) it follows that $r_T^{(\mathrm{min})} \leqslant r_T^{(o)}$ (cf. Veronis 1965). For $r_2^{(e)} > 0, r_T^{(\mathrm{min})} = r_T^{(e)}$. In the former case the steady *solution* will be called subcritical, while in the latter case it will be called supercritical.

The condition for marginal stability ($s = 0$) of the steady branch is

$$(1+a^2)^2[(\tau^2+a^2)^2+\tau r_S(\tau^2-a^2)] = r_T(1-a^2)(\tau^2+a^2)^2. \tag{23}$$

By substituting for r_T from (21) we can reduce equation (23) to

$$a^2[(\tau^2+a^2)^2+\tau r_S(\tau^2-1)] = 0. \tag{24}$$

The root $a^2 = 0$ corresponds to the neutrally stable point $r_T^{(e)}$, while the other root is simply the condition for a turning point at $r_T = r_T^{(\mathrm{min})}$, where $r_T^{(\mathrm{min})}$ is given by (22). This result can be shown to be quite general (Poincaré 1885; Jeans 1928). For suppose that the equations of motion can be written in the form

$$\mathbf{X}' = \mathbf{F}(\mathbf{X}, r_T), \tag{25}$$

where $\mathbf{X}^T = (a, b, c, d, e)$. Let $\mathbf{X}_0$ be a steady solution such that $\mathbf{F}(\mathbf{X}_0, r_T) = 0$ and let $\boldsymbol{\xi} = \mathbf{X}-\mathbf{X}_0$. Then the equations describing the linear stability of the steady solution are

$$\boldsymbol{\xi}' = \left(\frac{\partial \mathbf{F}}{\partial \mathbf{X}}\right)_0 \boldsymbol{\xi}, \tag{26}$$

with the corresponding dispersion relation

$$\det\left[s\delta_{ij} - \left(\frac{\partial F_i}{\partial X_j}\right)_0\right] = 0. \tag{27}$$

The neutrally stable points ($s = 0$) must therefore satisfy the condition

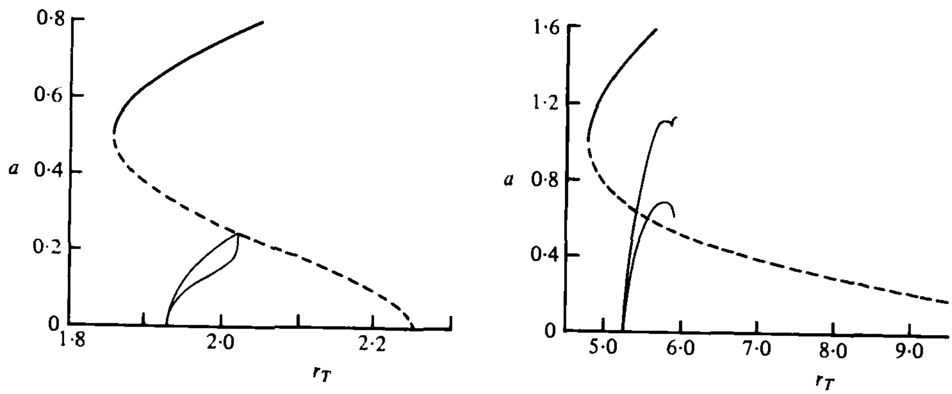
$$\det\left(\frac{\partial F_i}{\partial X_j}\right)_0 = 0. \tag{28}$$

However, for the steady solution

$$\left(\frac{\partial \mathbf{F}}{\partial \mathbf{X}}\right)_0 \delta\mathbf{X}_0 + \left(\frac{\partial \mathbf{F}}{\partial r_T}\right)_0 \delta r_T = 0. \tag{29}$$

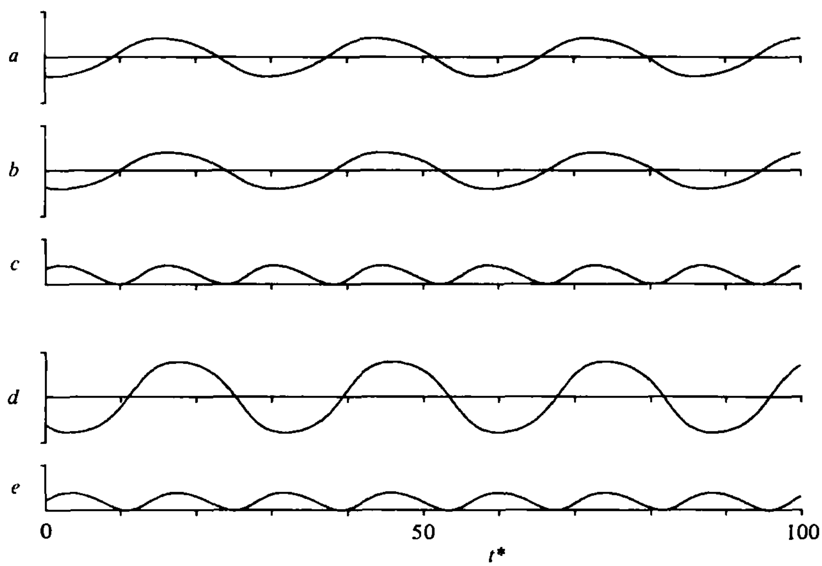
At a turning point a perturbation in the steady solution is characterized by $\delta r_T = 0$, $\delta \mathbf{X}_0 \neq 0$. It follows that the condition (28) has to be satisfied and that the point is therefore neutrally stable. This result depends on the roots of the dispersion relation being simple. Moreover, its converse is not true since neutrally stable points will also be found whenever $(\partial \mathbf{F}/\partial r_T)_0 = 0$ (i.e. at $r = r^{(e)}$).

We next consider the possibility of Hopf bifurcations characterized by the condition $\mathrm{Re}\, s = 0$. This condition yields an algebraic equation of sixth degree for a^2. There are



(a) (b)

FIGURE 1. The amplitude–Rayleigh-number diagram for (a) case A, (b) case B. The solid heavy line indicates the stable steady branch, the broken heavy line the unstable steady branch, and the solid thin lines denote the maximum and root mean square values of a on the oscillatory branch.



(a)

Figure 2(a). For the caption see next page.

no large positive roots except for a possible root, present if σ is sufficiently large, that corresponds to the bifurcation leading to aperiodic solutions of the Lorenz (1963) type. It can also be shown that the Lorenz root is the only positive root when $\tau \ll 1$. We have found no evidence for the presence of any other Hopf bifurcations for other values of the parameters. Since the model is not expected to be valid for the large values of r_T corresponding to the Lorenz root we shall not discuss this bifurcation further. The properties of the steady branch can now be summarized as follows:

(i) $\omega_0^2 < 0$. If $r_2^{(e)} > 0$ the branch of steady solutions is supercritical and therefore stable (Hopf 1942), except for a possible bifurcation of the Lorenz type. The subcritical case with $r_2^{(e)} < 0$ can typically be obtained from the supercritical one by decreasing τ.

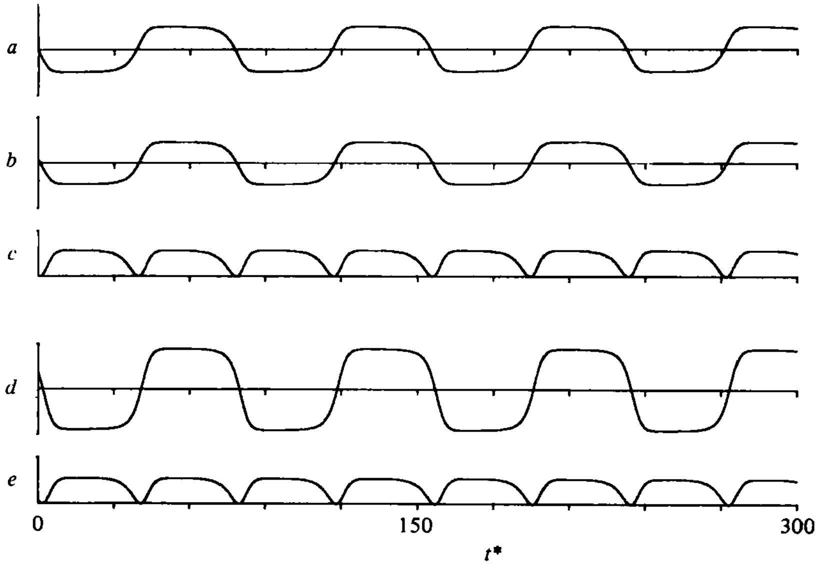


FIGURE 2. Typical sets of oscillatory solutions for case A with (a) $r_T = 2{\cdot}0$, (b) $r_T = 2{\cdot}0219$. The modes c and e, which are related to the thermal and solutal Nusselt numbers, have half the period of the other modes. The half-lengths of the ordinate axes are 0·5 for a, b, d, e and 0·1 for c.

Since an eigenvalue can change sign at $r_T^{(\min)}$ only, and there are no other Hopf bifurcations, it follows from the continuous dependence of the eigenvalues on τ that the upper steady branch is again stable. Hence the positive eigenvalue associated with the unstable subcritical branch first increases from zero at $r_T^{(e)}$ and then decreases, passing through zero at $r_T^{(\min)}$.

(ii) $\omega_0^2 > 0\,(r_T^{(o)} < r_T^{(e)})$ and $r_2^{(e)} < 0$. This is the case of interest and is illustrated in figure 1. Note that only the branch of positive solutions is shown. This case can be obtained from (i) by increasing σ. As before it follows that the upper steady branch is stable, so that the large eigenvalue associated with the instability of the subcritical branch near $r_T^{(e)}$ has to pass through zero at $r_T^{(\min)}$.

All the above properties are in excellent qualitative agreement with the numerical solutions of the full problem (Veronis 1968; Huppert & Moore 1976). We note, in particular, that the model supports the conjecture of Huppert & Moore that for $\tau < 1$ the stable upper branch develops into an unstable subcritical branch which bifurcates from $r_T^{(e)}$.

5. The oscillatory solutions

The branch of oscillatory solutions bifurcates from $r_T^{(o)}$. In the context of the model $r_T^{(o)} \geqslant r_T^{(\min)}$, although this is not in general true of the full equations. (The truncation favours subcritical convection.) Nonlinear oscillations cannot be studied analytically and we have therefore integrated the equations (11) numerically, as an initial-value problem, using a fourth-order Runge–Kutta–Merson scheme with appropriately

Case	r_S	σ	τ	ϖ
A	0·5	10	0·4	8/3
B	4	10	0·4	8/3
C	6	10	0·4	8/3

TABLE 1. Parameters for cases considered in this paper.

chosen timesteps. Linear theory shows that the value of R_T at the onset of instability (whether by overstable or direct modes) is least when $\lambda = 2^{\frac{1}{2}}$ ($\varpi = \frac{8}{3}$). Here we shall confine our attention to calculations that are supported by the results of two-dimensional numerical experiments, and follow Veronis (1968) and Huppert & Moore (1976) in setting $\varpi = \frac{8}{3}$ for all our computations.

The numerical results provide values of a, b, c, d, e as functions of t^*. We specify the amplitude of the oscillation by the maximum of $|a(t^*)|$ once the solution has settled to a periodic state. The oscillatory branch can then be represented by plotting the amplitude as a function of r_T. In this paper we describe three cases, differing only in the values of r_S; the parameters for these cases are specified in table 1. Figure 1(a) summarizes the results for case A. The bifurcation is supercritical and the amplitude

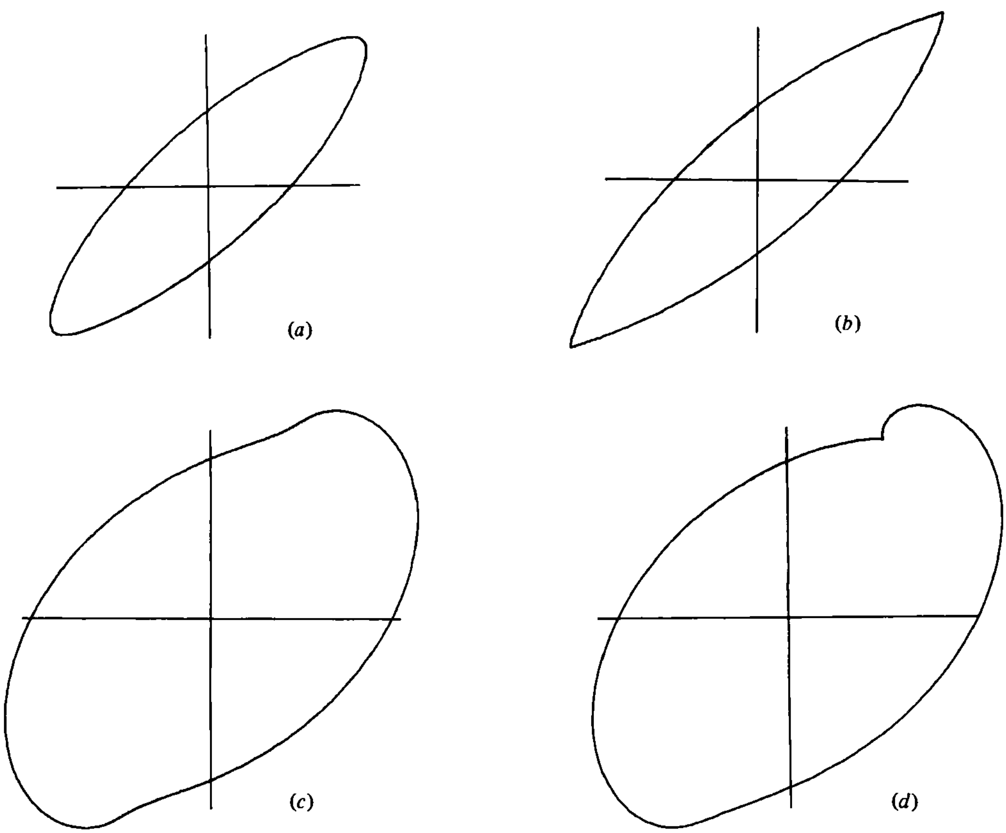


FIGURE 3. Limit cycles projected onto the ad plane. Case A (a) $r_T = 2·0$, (b) $r_T = 2·0219$. Case B (c) $r_T = 5·853$, (d) $r_T = 5·8722$. The axes have half-lengths 0·2, 0·4 for case A and 1·0, 0·5 for case B.

increases monotonically with r_T until the branch reaches the subcritical unstable branch of the steady solutions at $r_T = r_T^{(c)}$, where the oscillations disappear and the solution jumps to the stable upper branch. If r_T is now decreased again the solution will remain steady until $r_T = r_T^{(\min)}$ where it drops to the conductive solution. Thus throughout the range $r_T^{(o)} < r_T < r_T^{(c)}$ both stable steady solutions and stable limit cycles are possible and hysteresis can occur. However, the limit cycles are 'less stable' than the steady solutions, for a sufficiently large perturbation will send them to the upper steady branch.

The system (11) is invariant under a transformation that changes the signs of a, b and d, and leaves c and e unchanged. Thus if P is the time for a, b or d to complete a full oscillation, the modes c and e, which are related respectively to the thermal and solutal Nusselt numbers, will have periods $\frac{1}{2}P$. This is illustrated in figure 2, where two typical sets of solutions are shown, one for r_T not far from $r_T^{(o)}$ and the other for r_T close to $r_T^{(c)}$. The corresponding limit cycles, projected onto the ad-plane, are shown in figure 3. As r_T approaches $r_T^{(c)} \approx 2{\cdot}02195$, the profiles flatten and the limit cycle becomes cigar-shaped. In the neighbourhood of $r_T^{(c)}$ the period increases rapidly, and as $r_T \rightarrow r_T^{(c)}$ it becomes infinite, although the amplitude shows little variation (figure 4). This behaviour can be better understood by considering a phase portrait of the solution at some fixed value of $r_T, r_T^{(o)} < r_T < r_T^{(c)}$, as sketched in figure 5 (a). The oscillatory solution about the static solution (which is unstable to either an oscillatory or a direct mode depending on the value of r_T) is symmetrical, and corresponds to a limit cycle lying between two symmetrically spaced saddle points. At still larger amplitudes lie the symmetrically placed stable fixed points corresponding to steady solutions on the upper branch. Distant trajectories can wind round both fixed points several times, corresponding to decaying large-amplitude oscillations, before spiralling into one or other of the fixed points. As $r_T \rightarrow r_T^{(c)}$ the separatrices passing through the saddle points connect, forming a heteroclinic limit cycle joining these points as indicated in figure 5 (b) (cf. figure 3b); the period of this orbit is infinite. The final sketch, in figure 5 (c), represents trajectories for $r_T^{(c)} < r_T < r_T^{(e)}$; the limit cycle no longer exists and all trajectories ultimately converge on one of the two stable fixed points. This behaviour has some resemblance to the finite-amplitude motion of a simple pendulum. Indeed, Knobloch & Proctor (1981) have shown that in the limit $0 < \omega_0^2 \ll 1$ the solutions can be obtained analytically in terms of elliptic functions, and have confirmed that the period becomes infinite as $r_T \rightarrow r_T^{(c)}$.

The main effect of increasing r_s is an increase in the amplitude of the oscillations, which can now be substantially larger than $a(r_T)$ on the unstable steady branch. As a function of time, $a(t^*)$ overshoots its value on the steady branch but the profile develops an increasingly long plateau, near that value, as $r_T \rightarrow r_T^{(c)}$. In the limit cycles this behaviour is reflected in the appearance of a pair of cusps near the saddle points. For larger values of r_s the transition between the oscillatory and steady branches proceeds in a qualitatively different manner. The amplitude–Rayleigh-number diagram for case B is shown in figure 1 (b). As r_T is increased, the amplitude rises rapidly and the oscillations cease to resemble the linear solutions. The oscillatory branch reaches a maximum at $r_T \approx 5{\cdot}8$ and then begins to decrease with the solution $a(t^*)$ continuing to oscillate symmetrically about the static solution $a = 0$ until $r_T \approx 5{\cdot}855$, where the solution bifurcates to *asymmetrical* oscillations, in which $a(t)$ spends more time on one side of $a = 0$ than on the other. The period P does not change at this

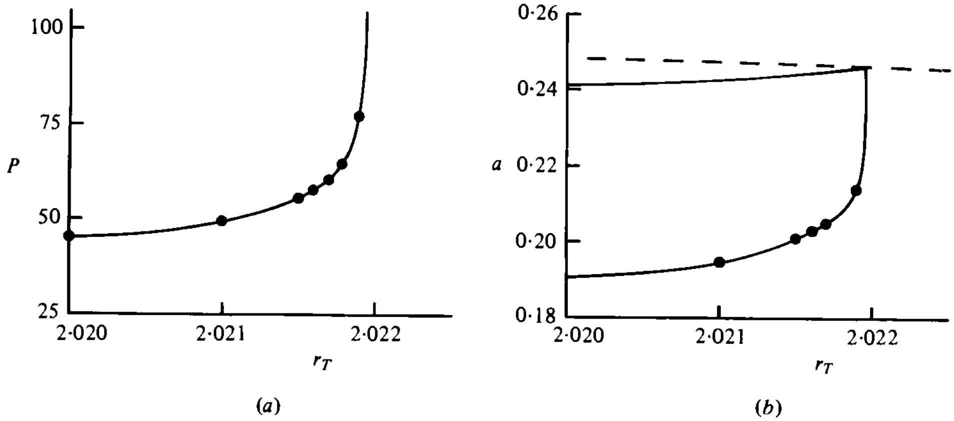


FIGURE 4. (*a*) The period P as a function of r for case A. Note the rapid increase near $r_T^{(c)} \approx 2{\cdot}02195$. (*b*) The amplitude of the oscillation in the same range of r. The upper and lower solid lines denote the maximum and root mean square values of $a(t^*)$ respectively; the broken line indicates the unstable steady branch. Near $r_T^{(c)}$ both the maximum and the r.m.s. value approach the steady branch, as the oscillations spend an increasingly large proportion of their period near the maximum amplitude.

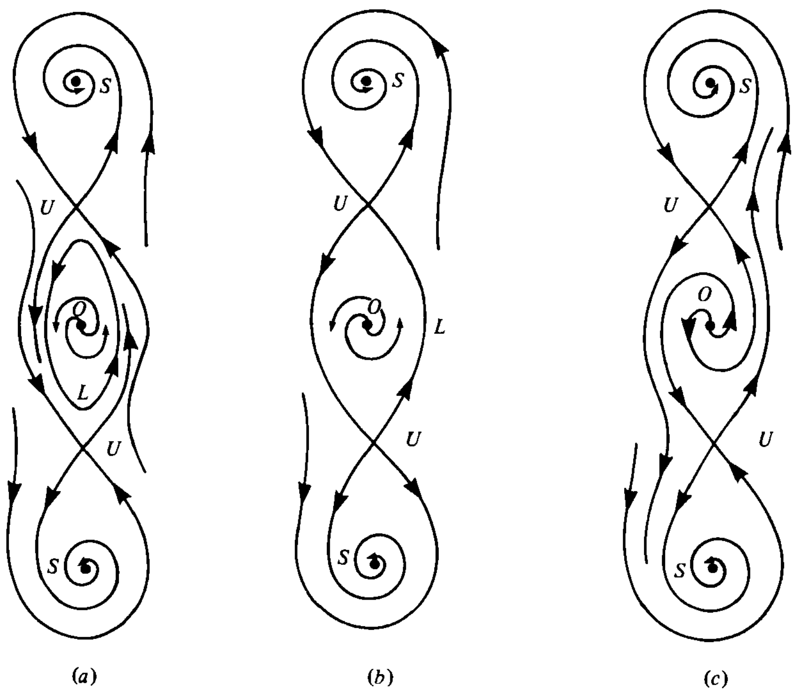


FIGURE 5. Sketches of the solution trajectories projected onto the a'–a plane for (*a*) $r_T^{(o)} < r_T < r_T^{(c)}$, (*b*) $r_T = r_T^{(c)}$ and (*c*) $r_T^{(c)} < r_T < r_T^{(e)}$. O denotes the static solution (0, 0), U the unstable steady solutions and L the limit cycle.

bifurcation although, since c and e are rectified forms of a, the bifurcation is manifested by a doubling of their periods from $\frac{1}{2}P$ to P. Figure 6 shows the set of oscillations just before and shortly after this bifurcation. The corresponding limit cycles are displayed in figure 3(*c*) and (*d*). The asymmetry is discernible in the profile of $a(t^*)$

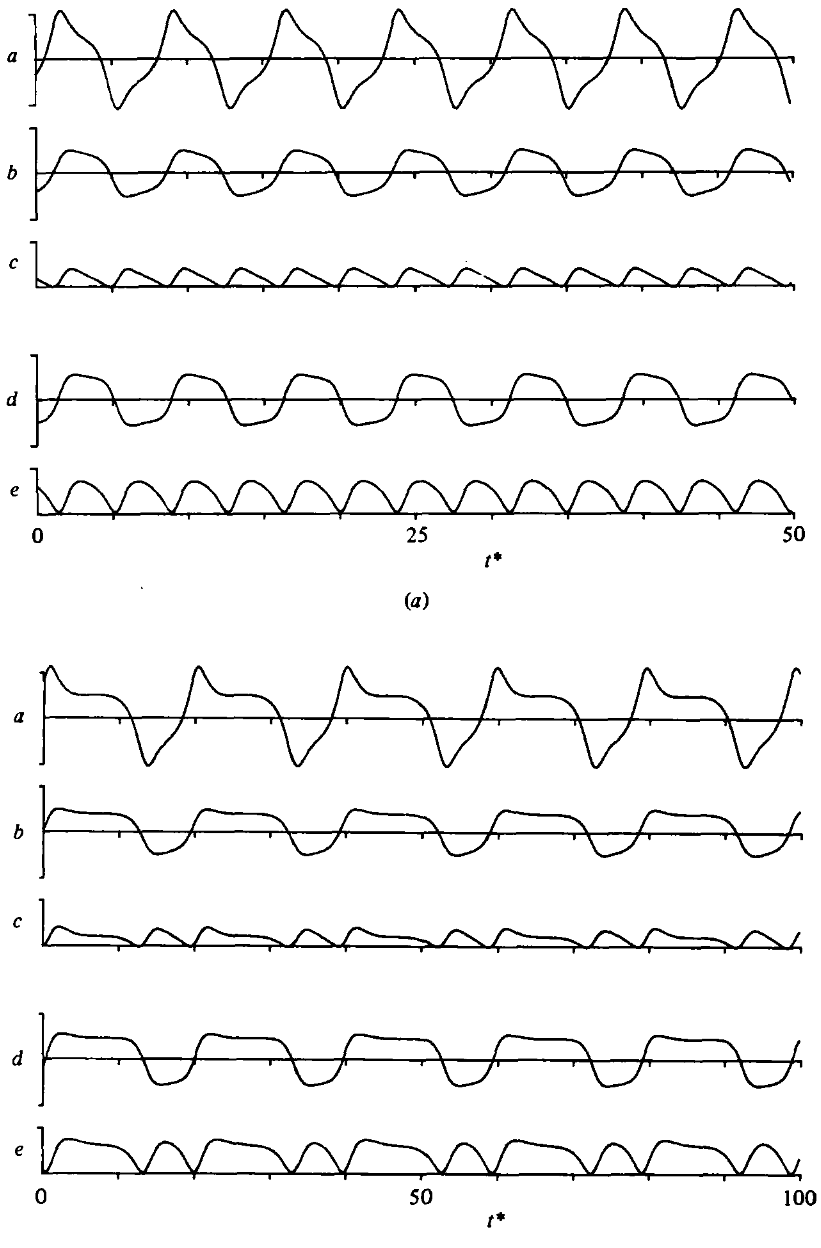


FIGURE 6. As figure 2 but for case B with (a) $r_T = 5{\cdot}853$ (symmetrical oscillations), (b) $r_T = 5{\cdot}8722$ (asymmetrical oscillations). Note that the asymmetry in modes a, b and d manifests itself as an apparent period doubling in modes c and e. The half-lengths of the ordinate axes are $1{\cdot}0$ for a, b, d, e and $0{\cdot}1$ for c.

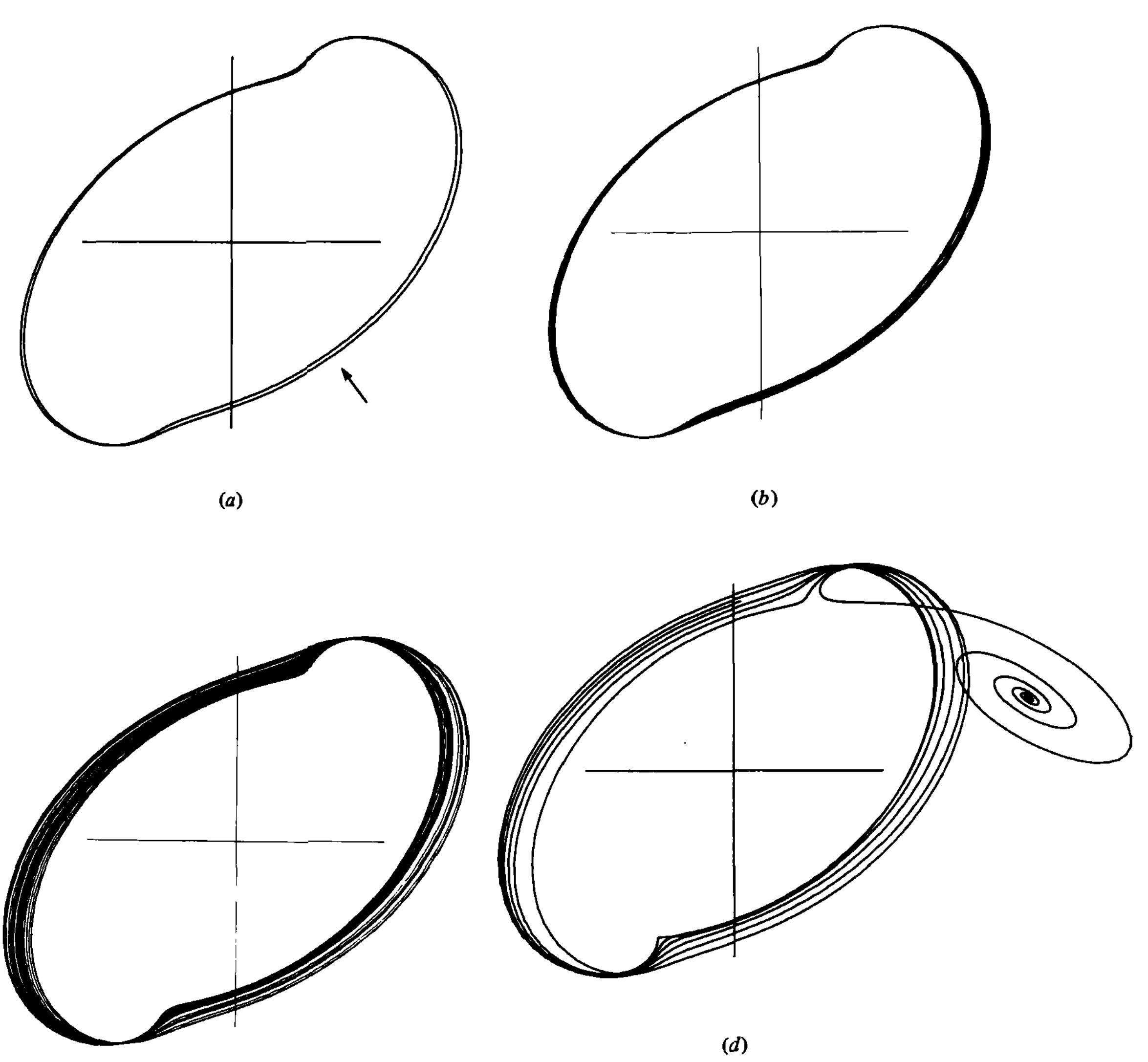
in figure 6(*b*) and quite apparent in $c(t^*)$ and $e(t^*)$. The corresponding limit cycle develops a marked bump in the first quadrant; there is, of course, another solution which is a mirror image of this one. As r_T is increased the asymmetry becomes more prominent. By $r_T = 5{\cdot}8723$ a small loop develops in the neighbourhood of the unstable fixed point. At $r_T = 5{\cdot}8724$ the oscillatory solution has lost stability and the trajectory spirals in to the stable fixed point.

Case C, with a higher value of r_s, exhibits more exotic behaviour. The bifurcation to asymmetry, at $r_T \approx 7{\cdot}785$, is followed by a bifurcation at which the period doubles, at $r_T \approx 7{\cdot}793$. The corresponding limit cycle is depicted in figure 7(*a*) and the region indicated is enlarged in figure 8(*a*). Successive cycles traverse the two segments alternately. At $r_T \approx 7{\cdot}7949$ there is a further period-doubling bifurcation that results in the appearance of four segments, as shown in figure 8(*b*). The trajectory describes these segments in the order indicated and the solution repeats itself exactly after four cycles. Further period-doubling bifurcations result in the solution repeating itself after 2^n cycles ($n = 3, 4, \ldots$). We have located bifurcations at $r_T \approx 7{\cdot}7953$ ($n = 3$), $r_T \approx 7{\cdot}79546$ ($n = 4$) and $r_T \approx 7{\cdot}79550$ ($n = 5$). Details of solutions with $n = 3, 4$ are shown in figure 8(*c*) and (*d*). For $r_T > 7{\cdot}7955$ we have found no more periodic solutions and the oscillations are apparently aperiodic.

Such aperiodic solutions are shown in figure 7(*b*), (*c*) for $r_T = 7{\cdot}796$, $7{\cdot}798$ respectively. In the former the trajectory wanders in the neighbourhood of an (unstable) asymmetrical solution. This neighbourhood is apparently well defined, although the distribution of trajectories within it is highly non-uniform, as shown in figure 8(*e*). In the latter example the trajectory switches, apparently at random, from the neighbourhood of one asymmetrical solution to that of its mirror image. However, the trajectory possesses not just a double but a multiply braided structure, as shown in figures 7(*c*) and 8(*f*). We expect that there exists a hierarchy of such structures, as in the example studied by Marzec & Spiegel (1980).

These aperiodic solutions extend over a finite range of r_T but when $r_T > 7{\cdot}8$ the attractor loses its stability and the trajectory escapes to the stable fixed point. Figure 7(*d*) shows a solution for $r_T = 7{\cdot}8$: the trajectory describes a number of cycles before escaping and spiralling into the fixed point. The loss of stability of the attractor is apparently associated with the development of a cusp (visible in the third quadrant) in the neighbourhood of either unstable fixed point. We conjecture that this occurs when the oscillatory branch doubles back, and that its unstable continuation terminates on the unstable steady branch as before.

These phenomena are complicated by transient behaviour. The oscillations become extremely sensitive to small changes in r_T so that transients take a long time to decay. Marzec and Spiegel (1980) show a remarkable example of prolonged transient behaviour leading eventually to a doubly periodic limit cycle. Hence it may be necessary to integrate for many periods before the periodicity becomes apparent. For instance, the examples in figure 8(*d*)–(*f*) were integrated up to $t^* \approx 500$, covering about 100 cycles. Moreover, the limit cycles have a restricted domain of attraction in this region. If r_T is increased too rapidly, or the initial conditions are inappropriately chosen, the solution spirals to a fixed point on the stable steady branch.



(*a*) (*b*) (*d*)

FIGURE 7. Limit cycles for case C, projected onto the ad-plane. (*a*) Period doubling for $r_T = 7{\cdot}7948$. Chaotic solutions for (*b*) $r_T = 7{\cdot}796$, (*c*) $r_T = 7{\cdot}898$. (*d*) $r_T = 7{\cdot}8$: the attractor has lost stability and the trajectory spirals into a fixed point. The axes have half-lengths 1·0, 0·5.

6. Discussion

The results obtained above correspond closely to those found by Huppert & Moore (1976) for the full system defined by (3)–(7), apart from quantitative details. For moderate values of r_s the transition from oscillations to steady motion takes place by the period of oscillations going infinite, and the oscillatory branch terminates on the unstable steady branch. For a case with $r_s = 4{\cdot}81$, $\sigma = 1$, $\tau = 0{\cdot}1$, Huppert & Moore found a transition to asymmetrical oscillations at $r_T \approx 5{\cdot}1$ and interpreted this as a period doubling, owing to the apparent change in period of the Nusselt numbers (see figure 5(*d*) of their paper). We have seen that the true period P does not alter at such a bifurcation. Later on, at $r_T \approx 5{\cdot}5$, they obtained aperiodic solutions which persisted

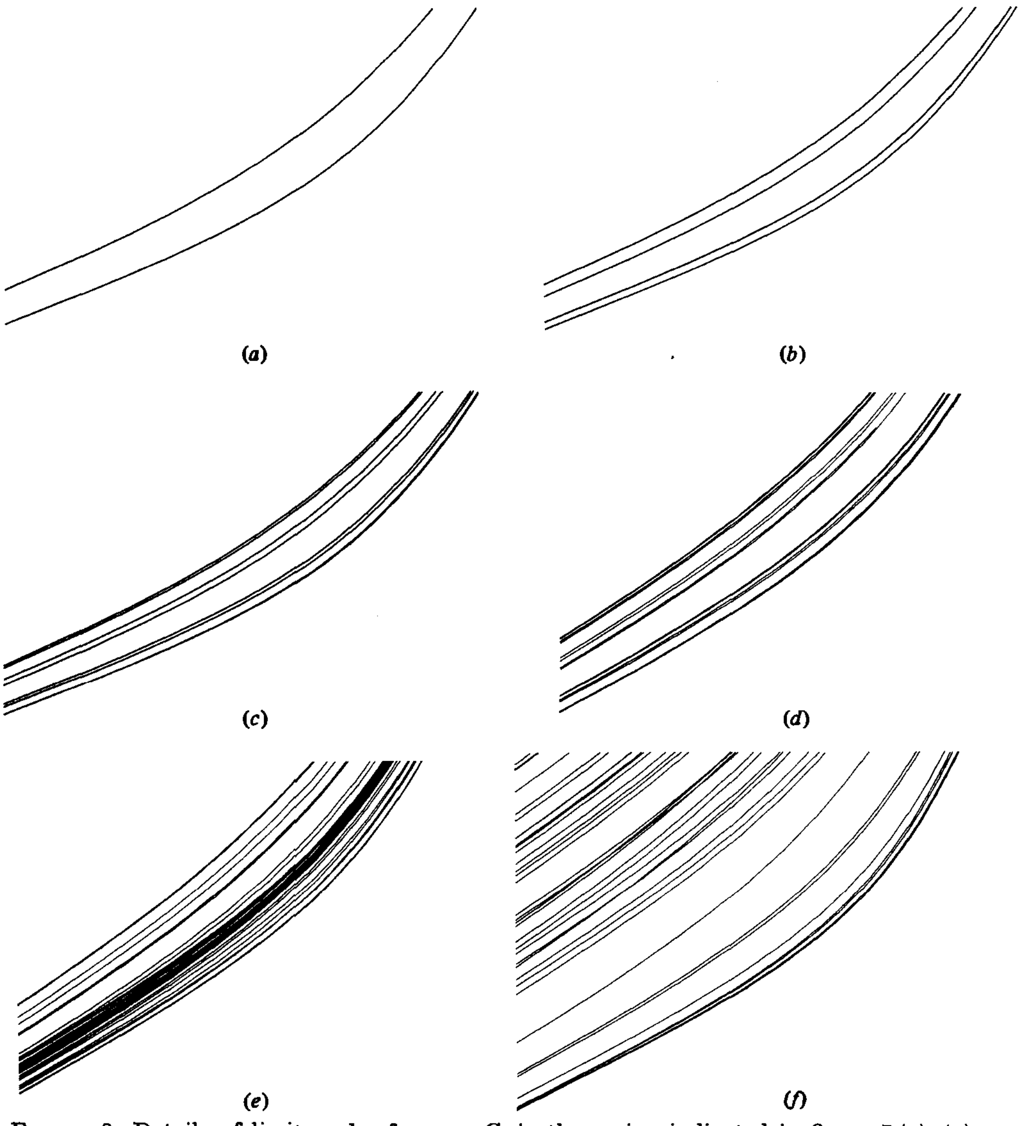


FIGURE 8. Details of limit cycles for case C, in the region indicated in figure 7 (*a*). (*a*) $r_T = 7{\cdot}7948$ (doubling), (*b*) $r_T = 7{\cdot}7952$ (quadrupling), (*c*) $r_T = 7{\cdot}79545$ (octupling), (*d*) $r_T = 7{\cdot}79548$ (16-fold), (*e*) $r_T = 7{\cdot}796$ (aperiodic), (*f*) $r_T = 7{\cdot}798$ (aperiodic). The scale for (*f*) differs from that for (*a*)–(*e*).

until $r_T \approx 6{\cdot}1$. However, solutions to equations (11) with their choice of parameters do not exhibit similar behaviour. Nevertheless, our results suggest that in the full problem a sequence of bifurcations with true successive doublings of the period was masked by apparent aperiodicity. In order to detect the first two doublings of P we used a small but variable time step and integrated, using a fourth-order scheme, for more than 50 periods. This is possible for ordinary, but not for partial, differential equations.

The bifurcation pattern that we have described here is similar to the pitchfork pattern found for the logistic difference equation (May 1976). At each successive bifurcation the limit cycle becomes unstable, shedding two stable orbits with twice its period; this is manifested in the splitting of the trajectory, cf. figure 8(*a*)–(*d*). Although we have not succeeded in locating more than 5 such bifurcations, we believe that there is an infinite sequence of pitchfork bifurcations, at each of which the period doubles, until it becomes infinite at some finite value of r_T. This resembles the behaviour of solutions to difference equations (May 1976) as well as those to various third- and fifth-order autonomous systems. Baker, Moore & Spiegel (1971) studied a model of overstable convection in which the transition to asymmetrical oscillations occurred when a stability parameter $\Lambda \approx 0{\cdot}57$ and was followed by successive period doublings in the range $0{\cdot}620 \leqslant \Lambda \leqslant 0{\cdot}622$ (Marzec & Spiegel 1980) after which the solution became aperiodic. As the Rayleigh number is decreased, the solutions of the Lorenz equations (Shimizu & Morioka 1978; Morioka & Shimizu 1978; Franceschini 1980) also show a bifurcation to asymmetry followed by successive period doubling until chaotic solutions are obtained.

In such systems, if $r_T^{(n)}$ is the value of r_T at which the nth doubling occurs, the location of successive bifurcations may be described in terms of the ratio

$$\delta_{n-1} = \frac{r_T^{(n)} - r_T^{(n-1)}}{r_T^{(n+1)} - r_T^{(n)}}. \tag{30}$$

Feigenbaum (1978) has shown for a wide class of difference equations that δ_n tends asymptotically to a value 4·669 2... for large n. It has been conjectured (e.g. Pomeau 1977) that the same universal constant applies to differential equations too. Our results for case C are consistent with this conjecture. Marzec & Spiegel (1980) found, for a third-order system of differential equations, that $\delta_1 = 6{\cdot}9$ and $\delta_2 = 4{\cdot}8$. Franceschini (1980) established, for a particular sequence of bifurcations in the Lorenz system, that $\delta_1 = 2{\cdot}32$, $\delta_2 = 4{\cdot}32$, $\delta_3 = 4{\cdot}58$, $\delta_4 = 4{\cdot}67$. Similar results have been reported for a fifth-order system (Boldrighini & Franceschini 1979; Franceschini & Tebaldi 1979). These calculations all suggest that the location of successive period doublings is asymptotically universal. Moreover, for the present model such bifurcations were preceded by a bifurcation to asymmetrical solutions, which is a necessary prerequisite for period doubling. Beyond the accumulation point of these bifurcations, estimated from $\delta_n \approx 5$, we have found only aperiodic solutions despite integrating for 150 cycles. Collectively the above results strongly suggest that the model equations contain a strange attractor. This attractor is present at much smaller Rayleigh numbers than the Lorenz attractor, in a regime where the model provides a qualitatively accurate indication of the behaviour of solutions to the full two-dimensional problem.

We believe, therefore, that there is a narrow range of R_T containing an infinite sequence of pitchfork bifurcations for the full problem too, though we doubt whether it is possible to find it. The only means of investigating such a transition seems to be by solving a simplified problem like that which we have considered here. Since thermosolutal convection is the paradigm for a range of doubly diffusive configurations, we expect that similar solutions exist for all those problems too. Finally, it should be pointed out that, although it is of great interest to establish the nature of these transitions for solutions to the relevant equations, we do not expect that doubling could be demonstrated in any real system. As in the numerical experiments, aperiodicity would gradually become more marked and it would be difficult to distinguish 'preturbulence' from actual chaos.

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